BOUNDS FOR THE PRINCIPAL FREQUENCY OF NONUNIFORMLY LOADED STRINGS*

BY

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ABSTRACT

n particles (beads) of different masses are mounted at equally spaced points along a string which in itself is weightless, of unit tension and fixed at its endpoints. Given the set of masses, it is shown that the principal frequency of this loaded string becomes minimal if they are arranged decreasing from the center in as nearly a symmetrical order as possible. If a strictly symmetrical increasing arrangement of the masses exists, then this arrangement gives the maximum of the principal frequency.

1. **Introduction.** In a paper by P. R. Beesack and the author [1] differential systems corresponding to strings with continuous, but nonhomogeneous, density $p(x)$ were considered. Setting the constant tension of the string equal to 1 and fixing it at its endpoints $x = 0$ and $x = L$, the square of the principal frequency is the least characteristic value $\lambda_1 = \lambda_1(p)$ of

$$(1.1) \quad y''(x) + \lambda p(x)y(x) = 0, \quad y(0) = y(L) = 0; \quad (p(x) > 0, \quad 0 \leqq x \leqq L).$$

Together with the given density $p(x)$ all densities equimeasurable to it in $[0, L]$ were considered. It was proved that among all strings of such a class the least principal frequency occurs in the case that the density is symmetrically decreasing about the midpoint of the string and the maximum principal frequency occurs when the density is symmetrically increasing. These extremizing densities, the symmetrically decreasing and increasing rearrangement of $p(x)$, were denoted by $p^-(x)$ and $p^+(x)$ respectively. The result just described was hence formulated as

$$(1.2) \qquad \lambda_1(p^-) \leqq \lambda_1(p) \leqq \lambda_1(p^+),$$

[1, Theorem 2] and it is easily seen that the corresponding equality holds only if $p = p^-$ or $p = p^+$, respectively [See 8, Theorem 1].

In the present paper we consider the analogous question for the discrete case. n particles (beads) of different masses are mounted at equally spaced points along a string, which in itself is weightless and of unit tension. The string is again fixed at

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its endpoints $x = 0$ and $x \doteq L$ and the mass m_i is attached at $x_i = iL/(n+1)$, $i = 1, \ldots, n$. Let y_i be the amplitude of the ith particle for a harmonic oscillation of this loaded string. The column vector $y = (y_1, \ldots, y_n)$ satisfies

$$A_2 y = \lambda P y, \tag{1.3}$$

where $A_2 = A_2^{(n)}$ is the Jacobi (tri-diagonal) matrix of order n:

$$A_2^{(n)} = A_2 = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}, \tag{1.4}$$

λ is the square of the frequency, and

$$P = \{p_1, \ldots, p_n\} \tag{1.5}$$

is the diagonal matrix of order n whose elements are $p_i = m_i L/(n+1)$ $i = 1, \ldots, n$ [4, Chapter III, and 6, Chapter 3]. (As in future only the p_i's and not the m_i's appear, we shall refer to p_i as the mass of the ith bead). The square of the principal frequency is the least characteristic value $\lambda_1 = \lambda_1(P)$ of the pencil $A - \lambda P$; i.e. $\lambda_1(P)$ is the smallest root of the characteristic equation $|A - \lambda P| = 0$.

Our problem is the following: Given a set of beads, how are they to be arranged at the equidistant points x_i in order to extremize λ_1? In contradistinction to the continuous case, there exists in general no arrangement which is strictly symmetrical with regard to the midpoint $x = L/2$ of the string. Nevertheless, the analogue of $\lambda_1(p^-) \leqq \lambda_1(p)$ holds in the following sense: if the beads are arranged in as nearly symmetrically decreasing order as possible, with the inevitable overweight given at each step to one and the same side, then λ_1 takes its least possible value. This is essentially Theorem 1 of §3; however, we formulate this result in a slightly more general way by replacing A_2 by A_k (see (2.6)).

This theorem is the main result of our paper. §2 contains the necessary definitions and two lemmas. Lemma 2 is of special importance for the proof of Theorem 1. Its first part follows from a theorem of Hardy, Littlewood and Pólya [5, Theorem 371]; its second part, needed for a uniqueness statement in Theorem 1, seems to be new. Both parts of this lemma follow from a result on rearrangements proved by A. Lehman [7].

In §4 an inequality corresponding to $\lambda_1(p) \leqq \lambda_1(p^+)$ is established but only for the case where a stricly symmetrically increasing arrangement of the beads exists.

2. **Definitions, notation and two lemmas.** We start with the following definitions which are slightly different from those used in the book of Hardy, Littlewood, and Pólya [5, Chapter X].

An ordered set $(a) = (a_1, \dots, a_n)$ of n real numbers is called *symmetrically decreasing* if either

$$a_1 \leqq a_n \leqq a_2 \leqq a_{n-1} \leqq \dots \leqq a_{[(n+2)/2]} \tag{2.1}$$

or

$$a_n \leqq a_1 \leqq a_{n-1} \leqq a_2 \leqq \dots \leqq a_{[(n+1)/2]} \tag{2.2}$$

holds. The set (a) is *symmetrically increasing* if either

$$a_1 \geqq a_n \geqq a_2 \geqq a_{n-1} \geqq \dots \geqq a_{[(n+2)/2]} \tag{2.3}$$

or

$$a_n \geqq a_1 \geqq a_{n-1} \geqq a_2 \geqq \dots \geqq a_{[(n+1)/2]} \tag{2.4}$$

holds. The set (a) is *strictly symmetrically increasing* if

$$a_1 = a_n \geqq a_2 = a_{n-1} \geqq \dots \geqq (=) a_{[(n+2)/2]}. \tag{2.5}$$

For a given set $(p) = (p_1, \dots, p_n)$ there exist, in general, two distinct, *symmetrically decreasing rearrangements.* The rearrangement ordered as in (2.1) is denoted by $(p^-) = (p_1^-, \dots, p_n^-)$ so that

$$p_1^- \leqq p_n^- \leqq p_2^- \leqq p_{n-1}^- \leqq \dots \leqq p_{[(n+2)/2]}^-; \tag{2.1'}$$

the other symmetrically decreasing rearrangement is denoted by $({}^-p) = ({}^-p_1, \dots, {}^-p_n)$:

$${}^-p_n \leqq {}^-p_1 \leqq {}^-p_{n-1} \leqq {}^-p_2 \leqq \dots \leqq {}^-p_{[(n+1)/2]}. \tag{2.2'}$$

Similarly, the *symmetrically increasing rearrangements* of (p) are denoted by (p^+) and $({}^+p)$ and their elements satisfy

$$p_1^+ \geqq p_n^+ \geqq p_2^+ \geqq p_{n-1}^+ \geqq \dots \geqq p_{[(n+2)/2]}^+ \tag{2.3'}$$

and

$${}^+p_n \geqq {}^+p_1 \geqq {}^+p_{n-1} \geqq {}^+p_2 \geqq \dots \geqq {}^+p_{[(n+1)/2]}. \tag{2.4'}$$

Finally, if (p) admits a *strictly symmetrically increasing rearrangement,* i.e. if $(p^+) = ({}^+p)$, then this rearrangement is denoted by $(p^*) = (p_1^*, \dots, p_n^*)$:

$$p_1^* = p_n^* \geqq p_2^* = p_{n-1}^* \geqq \dots \geqq (=) p_{[(n+2)/2]}^*. \tag{2.5'}$$

Sets (p) admitting such a rearrangement (p^*) are called *paired*. For even n a set (p) is paired if every value occurs an even number of times; for odd n the smallest value has to occur an odd number of times, every other value an even number of times.

If $(q) = (q_1, \ldots, q_n)$ is a rearrangement of $(p) = (p_1, \ldots, p_n)$ then we say that the diagonal matrix $Q = \{q_1, \ldots, q_n\}$ is a *rearrangement of the diagonal matrix* $P = \{p_1, \ldots, p_n\}$. Given P, its rearrangements P^-, ${}^-P$, P^+, ${}^+P$ and, if P is paired, P^* are defined by the order relations (2.1′)–(2.5′) for their elements.

Together with these diagonal matrices of order n, we consider also Jacobi matrices of the same order for which the elements in the diagonal are equal to the real constant k and the elements in the super- and sub-diagonal are -1. (All other elements are 0). We use the notation (cf. (1.4))

$$(2.6)\qquad A_k^{(n)} = A_k = \begin{bmatrix} k & -1 & & & & \\ -1 & k & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & -1 & k & -1 \\ & & & & -1 & k \end{bmatrix}.$$

($A_k^{(1)} = (k)$, $A_k^{(2)} = \begin{pmatrix} k & -1 \\ -1 & k \end{pmatrix}$ ...). For the corresponding quadratic forms the following two lemmas hold:

LEMMA 1. *Let*

$$(2.7)\qquad A_k^{(n)}(y,y) = A_k(y,y) = k\sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n-1} y_i y_{i+1}$$

be the quadratic form belonging to the matrix (2.6). (i) *If* $k > 2\cos(\pi/n+1)$ *then* $A_k(y,y)$ *is positive definite*; (ii) *if* $k < 2\cos(\pi/n+1)$ *then* $A_k(y,y)$ *takes negative values*; (iii) *if* $k = 2\cos(\pi/n+1)$ *then* $A_k(y,y)$ *is non-negative definite.*

This lemma is an immediate consequence of the following theorem due to Fan, Taussky and Todd [2, Theorem 9]. *If* $y_1, \ldots, y_n$ *are* n *real numbers, then*

$$\sum_{i=0}^{n} (y_i - y_{i+1})^2 > 4\sin^2\frac{\pi}{2(n+1)}\sum_{i=0}^{n} y_i^2$$

(*where* $y_0 = y_{n+1} = 0$) *unless* $y_i = c\hat{y}_i$, *where*

$$(2.8)\qquad \hat{y}_i = \sin\frac{i\pi}{n+1}, \qquad i = 1, \ldots, n.$$

LEMMA 2. *Let* $A_k(y,y)$ *be defined by* (2.7) *and let the set* (y) *of* n *non-negative numbers be given except in arrangement. Then* $A_k(y,y)$ *attains its minimum if*

(y) is arranged in symmetrically decreasing order. Moreover, if all the elements of (y) are positive and if no three elements of (y) have the same value, then $A_k(y, y)$ attains its minimum only if (y) is symmetrically decreasing.

In our notation, this can be stated as

$$A_k(y, y) \geqq A_k(y^-, y^-), \qquad y_i \geqq 0,\ i = 1, \ldots, n, \tag{2.9}$$

with the additional statement that, if $y_i > 0$, $i = 1, \ldots, n$ and if no three numbers y_i are equal, then equality holds in (2.9) only if $(y) = (y^-)$ or $(y) = (^-y)$.

We remark that $\sum_{i=1}^{n} y_i^2$ is invariant for all rearrangements of a given set. Defining

$$S(y, y) = \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} y_i y_{i+1}, \tag{2.10}$$

it follows that (2.9) is equivalent to

$$S(y, y) \leqq S(y^-, y^-). \tag{2.11}$$

This last inequality is a very special case of a Theorem of Hardy, Littlewood and Pólya on bilinear forms. [5, Theorem 371; to obtain (2.11) set, in their notation, $c_0 = 1$, $c_1 = c_{-1} = \frac{1}{2}$, all other $c = 0$; let their two sets (x) and (y) coincide and if (our) n is even, let one element of (y) be zero.] This proves the first part of Lemma 2. As mentioned, both parts of the lemma follow from Lehman's result. Indeed, if we apply the corollary of [7] to the function $f(x) = x^2$ we obtain that

$$\sum_{i=1}^{n-1} y_i y_{i+1}, \qquad y_i \geqq 0, \quad i = 1, \ldots, n,$$

is maximum if (y) is symmetrically decreasing and that under the more restrictive assumptions on (y) this maximum is attained only if (y) is symmetrically decreasing. This is clearly equivalent to Lemma 2.

3. The minimum of the least characteristic value.

THEOREM 1. *Let $A_k^{(n)} = A_k$ be the Jacobi matrix of order n defined by (2.6) and let $Q = \{q_1, \ldots, q_n\}$, $q_i > 0$, $i = 1, \ldots, n$ be any diagonal matrix of order n with positive elements. Denote the least characteristic value of the pencil $A_k - \lambda Q$ by $\lambda_1(Q)$. Let $P = \{p_1, \ldots, p_n\}$, $p_i > 0$, $i = 1, \ldots, n$ be a given diagonal matrix and let P^- and P^+ be its symmetrically decreasing and increasing rearrangement respectively.*

If

$$k > 2\cos\frac{\pi}{n+1}, \tag{i}$$

then

(3.1) $$\lambda_1(P) \geqq \lambda_1(P^-).$$

Moreover, if

(i′) $$2 \geqq k > 2\cos\frac{\pi}{n+1},$$

then equality holds in (3.1) *only if P itself is symmetrically decreasing.*

If

(ii) $$k < 2\cos\frac{\pi}{n+1},$$

then

(3.2) $$\lambda_1(P) \geqq \lambda_1(P^+).$$

Finally, if

(iii) $$k = 2\cos\frac{\pi}{n+1},$$

then

(3.3) $$\lambda_1(Q) = 0$$

for any diagonal matrix $Q = \{q_1, \ldots, q_n\}$, $q_i > 0$, $i = 1, \ldots, n$.

Proof. For given n and k let $y = (y_1, \ldots, y_n)$ be the characteristic (principal) column vector corresponding to the least characteristic value $\lambda_1(P)$ of the pencil $A_k - \lambda P$ [3, p. 310]. Let $|y| = (|y_1|, \ldots, |y_n|)$, then

$$A_k(y, y) = k\sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n-1} y_i y_{i+1} \geqq k\sum_{i=1}^{n} |y_i|^2 - 2\sum_{i=1}^{n-1} |y_i||y_{i+1}| = A_k(|y|, |y|),$$

and

$$P(y, y) = \sum_{i=1}^{n} p_i y_i^2 = \sum_{i=1}^{n} p_i |y_i|^2 = P(|y|, |y|).$$

It now follows from the minimum characterization of $\lambda_1(P)$ [3, p. 319] that $|y|$ is also a characteristic vector corresponding to $\lambda_1(P)$. But the recursive form of the n scalar equations of

(3.4) $$A_k y = \lambda_1(P) P y$$

shows that the characteristic vector belonging to $\lambda_1(P)$ is determined to within a scalar factor. We may therefore assume $y = |y|$ and it follows from (3.4) that $y > 0$, i.e. $y_i > 0$ for all i, $i = 1, \ldots, n$. [Cf. 4, p. 136].

Assume now that (i) holds. By Lemma 1, $A_k(y,y)$ is in this case positive definite, hence $\lambda_1(P) > 0$. Let y be the positive characteristic vector belonging to $\lambda_1(P)$ Then

$$\lambda_1(P) = \frac{A_k(y,y)}{P(y,y)} \geqq \frac{A_k(y^-,y^-)}{P^-(y^-,y^-)} \geqq \min_{x \neq 0} \frac{A_k(x,x)}{P^-(x,x)} = \lambda_1(P^-)(>0). \tag{3.5}$$

Here $y^- = (y_1^-, \ldots, y_n^-)$ is the symmetrically decreasing rearrangement of y:

$$(0<)\, y_1^- \leqq y_n^- \leqq y_2^- \leqq y_{n-1}^- \leqq \cdots \leqq y^-_{[(n+2)/2]}. \tag{3.6}$$

(3.6) and (2.1′) show that y^- and P^- are similarly ordered and it follows by a well-known theorem of Hardy, Littlewood and Pólya [5, Theorem 368] that

$$P(y,y) = \sum_{i=1}^{n} p_i y_i^2 \leqq \sum_{i=1}^{n} p_i^- y_i^{-\,2} = P^-(y^-,y^-). \tag{3.7}$$

(3.7) and the first part of Lemma 2 (i.e. (2.9)) imply the first inequality sign of (3.5). The minimum in the fourth term of (3.5) is taken over the class of all not identically vanishing vectors x, and $y^-(>0)$ clearly belongs to this class. By the minimum characterization of the least characteristic value, this minimum is $\lambda_1(P^-)$. This proves (3.5) and hence also (3.1).

For the next assertion of the theorem we have to apply the second part of Lemma 2. We already know that, for any k, the characteristic vector y corresponding to $\lambda_1(P)$ can be chosen in such a way that all its components y_i, $i = 1, \ldots, n$ are positive. It remains to be shown that, if (i′) holds, then no three components y_i have the same value. But in this case (3.4) implies (as $\lambda_1(P) > 0$, $p_i > 0$, $y_i > 0$, $i = 1, \ldots, n$)

$$A_k y > 0 \tag{3.8}$$

or, explicitly,

$$k y_i > y_{i-1} + y_{i+1}, \quad y_i > 0, \quad i = 1, \ldots, n, \quad (y_0 = y_{n+1} = 0). \tag{3.9}$$

(3.9) and (i′) give

$$2 y_i > y_{i-1} + y_{i+1}, \quad y_i > 0, \quad i = 1, \ldots, n, \quad (y_0 = y_{n+1} = 0). \tag{3.10}$$

i.e. $y > 0$ is strictly concave. This strict concavity implies that no three components y_i can have the same value and y thus fulfills the conditions required for the second part of Lemma 2.

Let k now be restricted to the range given by (i′) and assume that equality holds in (3.1), i.e.

$$\lambda_1(P) = \lambda_1(P^-). \tag{3.11}$$

The inequality signs in (3.5) thus become equality signs. As the minimum in the fourth term of (3.5) is obtained only for the characteristic vector of $A_k - \lambda P^-$ it follows that y^- is this vector:

$$A_k y^- = \lambda_1(P^-)P^-y^- . \tag{3.12}$$

Equality of the first two ratios of (3.5) implies (by (3.7) and the first part of Lemma 2, i.e. (2.9)) $A_k(y,y) = A_k(y^-,y^-)$. It thus follows from the second part of Lemma 2 that either

$$y = y^- \tag{3.13}$$

or

$$y = {}^-y. \tag{3.14}$$

Here ${}^-y = ({}^-y_1,\ldots,{}^-y_n)$ is the other symmetrically decreasing rearrangement of y, satisfying

$$(0<)\,{}^-y_n \leqq {}^-y_1 \leqq {}^-y_{n-1} \leqq {}^-y_2 \leqq \cdots \leqq {}^-y_{[(n+1)/2]}. \tag{3.15}$$

(3.4), (3.11), (3.12) (3.13) imply $P = P^-$. (2.1'), (2.2') and (3.6), (3.15) imply that ${}^-p_i = p^-_{n+1-i}$ and ${}^-y_i = y^-_{n+1-i}$, $i = 1,\ldots,n$. (3.12) gives thus

$$A_k {}^-y = \lambda_1(P^-)\, {}^-P\, {}^-y. \tag{3.16}$$

(3.4), (3.11), (3.14) and (3.16) imply $P = {}^-P$. Hence, if (i') holds, equality in (3.1) implies that P is symmetrically decreasing.

The proof of (3.2) is analogous to the proof of (3.1). If (ii) holds, then (by Lemma 1) $A_k(y,y)$ takes negative values and $\lambda_1(P)$ is thus negative. (3.5) is thus replaced by

$$(0>)\lambda_1(P) = \frac{A_k(y,y)}{P(y,y)} \geqq \frac{A_k(y^-,y^-)}{P^+(y^-,y^-)} \geqq \min_{x\neq 0}\frac{A_k(x,x)}{P^+(x,x)} = \lambda_1(P^+), \tag{3.17}$$

which follows by (2.9) and

$$P(y,y) = \sum_{i=1}^{n} p_i y_i^2 \geqq \sum_{i=1}^{n} p_i^+ y_i^{-2} = P^+(y^-,y^-). \tag{3.18}$$

(3.18) follows by [5, Theorem 368] as y^- and P^+ are oppositely ordered.

Finally, if (iii) holds, $\hat{y}$ defined by (2.8), is clearly the characteristic vector corresponding to $\lambda_1(Q) = 0$ for any $Q = \{q_1,\ldots,q_n\}$, $q_i > 0$.

This completes the proof of the theorem.

For $k = 2$ we gave the mechanical interpretation of this theorem in the introduction. For $k > 2$ the n beads should be connected by springs of equal strength (proportional to $k-2$) to the line $y = 0$ [See 4, p. 269 for the analogous continuous case]. In the critical case $k = 2\cos(\pi/n+1)$ the "negative" springs and the tension of the string result in a static equilibrium displacement whose shape ($\hat{y}_i = c \sin(i\pi / n + 1)$) is independent of the masses.

4. The maximum of the least characteristic value for $k = 2$ and paired P. We now restrict ourselves to the case $k = 2$. It follows that the least characteristic value $\lambda_1(Q)$ of $A_2 - \lambda Q$, ($Q = \{q_1, \ldots, q_n\}$, $q_1 > 0$, $i = 1, \ldots, n$) is positive. For the continuous string the proof of $\lambda_1(p^+) \geqq \lambda_1(p)$ was simpler than the proof of $\lambda_1(p) \geqq \lambda_1(p^-)$. For $n = 3$ (and $k = 2$) an easy computation shows that $\lambda_1(P^+) \geqq \lambda_1(P)$. But already for $n = 4$ this inequality is, in general, not valid. Indeed, for $P = \{p_1, p_2, p_3, p_4\}$ we obtain,

$$
\begin{aligned}
\phi(\lambda) &= |A_2^{(4)} - \lambda P| \\
&= 5 - 2\lambda(2p_1 + 3p_2 + 3p_3 + 2p_4) \\
&\quad + \lambda^2[3(p_1p_2 + p_1p_4 + p_3p_4) + 4(p_1p_3 + p_2p_3 + p_2p_4)] \\
&\quad - 2\lambda^3(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4) + \lambda^4 p_1p_2p_3p_4.
\end{aligned}
\tag{4.1}
$$

Note that only the coefficients of λ and λ^2 depend on the arrangement of P. Denote the four masses by a, b, c and d and assume that

$$0 < a < b < c < d. \tag{4.2}$$

Let

$$P = \{d, a, b, c\}. \tag{4.3}$$

Its symmetrically increasing rearrangement P^+ is then given by

$$P^+ = \{d, b, a, c\}. \tag{4.4}$$

Computing (4.1) for P and P^+ and subtracting we obtain (using an obvious notation)

$$\phi(\lambda) - \phi^+(\lambda) = (b - a)(d - c)\lambda^2. \tag{4.5}$$

(4.2) and (4.5) imply $\lambda_1(P) > \lambda_1(P^+)$.

Comparing $\phi(\lambda)$ of P (given by (4.3)) with the characteristic functions of all its rearrangements it can be shown that the maximum λ_1 corresponds to (4.3). But we did not succeed in finding the maximizing arrangement for $n > 4$ and general P. Indeed, it may well be that no such general maximizing arrangement exists and that the order relations defining the maximizing arrangement vary with the set of masses. However, for *paired* sets of masses the analogue of $\lambda_1(p^+) \geqq \lambda_1(p)$ is valid.

THEOREM 2. *Let $A_2^{(n)} = A_2$ be the Jacobi matrix of order n defined by* (1.4). *Let the diagonal matrix $P = \{p_1, \ldots, p_n\}$, $p_i > 0$, $i = 1, \ldots, n$ be paired and* *let* $P^* = \{p_1^*, \ldots, p_n^*\}$ *be its strictly symmetrically increasing rearrangement. Denote the least characteristic value of* $A_2 - \lambda P$ *and* $A_2 - \lambda P^*$ *by* $\lambda_1(P)$ *and* $\lambda_1(P^*)$ *respectively. Then*

$$\lambda_1(P^*) \geqq \lambda_1(P) \tag{4.6}$$

and equality holds only if $P = P^*$.

Proof. Let $y > 0$ be the characteristic vector of $A_2 - \lambda P^*$ belonging to $\lambda_1(P^*)$. Hence,

$$A_2 y = \lambda_1(P^*) P^* y. \tag{4.7}$$

By (2.5) P^* is "symmetric" in the sense that

$$p_i^* = p_{n+1-i}^*, \qquad i = 1, \ldots, n. \tag{4.8}$$

The characteristic vector y is determined to within a scalar factor and does not change signs. (4.7) and (4.8) imply therefore that

$$y_i = y_{n+1-i}, \qquad i = 1, \ldots, n. \tag{4.9}$$

This symmetry of y and its previous established concavity (3.10) imply that y is strictly symmetrically decreasing, i.e.

$$(0 <) y_1 = y_n < y_2 = y_{n-1} < \cdots y_{[(n+2)/2]}. \tag{4.10}$$

(4.10) and (2.5′) show that y and P^* are oppositely ordered. Theorem 368 of [5] gives

$$P^*(y, y) = \sum_{i=1}^{n} p_i^* y_i^2 \leqq \sum_{i=1}^{n} p_i y_i^2 = P(y, y). \tag{4.11}$$

Using (4.11) we obtain

$$\lambda_1(P^*) = \frac{A_2(y, y)}{P^*(y, y)} \geqq \frac{A_2(y, y)}{P(y, y)} \geqq \min_{x \neq 0} \frac{A_2(x, x)}{P(x, x)} = \lambda_1(P) \quad (> 0). \tag{4.12}$$

This proves (4.6). If

$$\lambda_1(P^*) = \lambda_1(P), \tag{4.13}$$

then the inequality signs in (4.12) become equality signs. As the minimum in the fourth term is obtained only for the characteristic vector of $A_2 - \lambda P$ it follows that y is this vector:

$$A_2 y = \lambda_1(P) P y. \tag{4.14}$$

(4.7), (4.13) and (4.14) give $P = P^*$. This completes the proof of the theorem.

We remark that Theorem 2 remains clearly correct if $A_2^{(n)}$ is replaced by $A_k^{(n)}$ where k satisfies condition (i′) of Theorem 1.

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